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Constantes de acoplamiento magnetoeléctricas en compuestos estratificados piezoeléctricos / piezomagnéticos
Magneto-electric coupling constants in piezoelectric / piezomaganetic layered composite

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Resumen

Durante los últimos años se han estudiado los composites piezoeléctricos / piezomagnéticos debido a las numerosas aplicaciones relacionadas con el acoplamiento entre estos materiales y los campos. En el presente trabajo se presentan dos modelos teóricos para el cálculo del factor de acoplamiento magneto / eléctrico del composite con conectividad 2-2. Utilizando el método de homogeneización asintótica, los coeficientes efectivos de un compuesto estratificado magneto-electro-elástico periódico se pueden obtener en forma de matriz. Mediante el uso de esta matriz se estudia un compuesto de dos capas formado por BaTiO3 y CoFe2O4 y se obtienen expresiones para los coeficientes efectivos. El factor de acoplamiento magneto / eléctrico efectivo en función de la fracción volumétrica piezoeléctrica se encuentra a partir de estos coeficientes particulares. Además, se analiza un modelo dinámico del composite piezoeléctrico / piezomagnético multicapa. El modelo dinámico se ha utilizado para determinar las constantes de acoplamiento magnetoeléctrico.
Palabras clave: elasticidad electromagnética (electroelasticidad, magnetoelasticidad); efectos magnetoeléctricos; constantes piezoeléctricas; materiales compuestos

PACS: 46.25.Hf; 75,85 + t; 77.65.Bn; 77.84.Lf

Abstract

During the last few years, piezoelectric/piezomagnetic composites have been studied due to the numerous applications related to the coupling between these materials and the fields. In the present work, two theoretical models for calculating the magneto/electric coupling factor of the composite with 2-2 connectivity, are presented. Using the asymptotic homogenization method, the effective coefficients of a periodic magneto–electro–elastic layered composite can be obtained in matrix form. By using this matrix, a two-layered composite formed by $BaTiO_3$ and $CoFe_2O_4$ are studied, and expressions for the effective coefficients are obtained. The effective magneto/electric coupling factor as a function of the piezoelectric volumetric fraction are found from these particular coefficients. In addition, a dynamic model of the multilayer piezoelectric/piezomagnetic composite is discussed. The dynamical model has been used to determinate the magnetoelectric coupling constants.

Keywords: electromagnetic elasticity (electroelasticity, magnetoelasticity); magnetoelectric effects; piezoelectric constants; composite materials

PACS: 46.25.Hf; 75.85.+t; 77.65.Bn; 77.84.Lf

Introduction

The study of materials that exhibit magneto-electric (ME) coupling has attracted a lot of interest due to the multiple applications related to these materials. ME coupling of laminate composites has been investigated under combined magnetic and mechanical loadings (Fang *et al.* 2013, 075009). In previous work, the ME effect in a three-phase ME composite is experimentally studied (Zeng *et al.* 2015, 11). In (Fua 2016, 1788), the authors analyzed the ME coupling in lead-free piezoelectric bilayer composites and ME phases. A five-phase laminate composite transducer based on nanocrystalline soft magnetic FeCuNbSiB alloy is presented; whose ME coupling characteristics have been investigated in (Qiu *et al.* 2014, 112401). (Zhou *et al*. 2017, 014016) where a strong ME coupling at the interface in a Co /$[PbMg_{1/3}Nb_{2/3}O_3]_{0.71}[PbTiO_3]_{0.29}$ bilayerd structure, was found. Composites with piezoelectric and piezomagnetic phases exhibit magnetoelectric properties due to the coupling of phases, and several researchers investigate the ME effect in these composites. (Kuo & Hsin 2018, 1503) investigated fibrous composites while (Shi 2018, 474), (Praveen 2018, 392) and (Hohenberger 2018, 184002) have studied laminate composites.

There are several ways to determine the coupling factors between different fields. In this paper, we have used two ways to determine the ME coupling factor. The first method is the asymptotic homogenization method. The effective coefficients are determined through the formulation of (Cabanas *et al.* 2010, 58). The double-scale asymptotic homogenization (MHA) method introduces two spatial coordinate systems: the local coordinate which studies the problems at the microstructure level, and the global coordinate system which uses the global characteristics of the composite. From these effective ME coefficients the coupling factor is obtained through the thermodynamic definition.

The second method was proposed in (Zhang & Geng 1994, 614) to determine the electro-elastic coupling factor (k_t) where a dynamic study of the laminate is performed. From the dispersion curves, the required parameters for calculating the ME coupling constant are calculated. Vertically polarized waves (SV waves) that propagate in the polarization direction of the materials are studied in each phase. Using contour equations at the interfaces of the composite, these waves can be related and can be obtained to the dispersion curves.

Homogenization methods are the most common type of methods, used for the calculation of coupling factors (Cabanas et al. 2010, 58). The asymptotic homogenization method, formally developed by (Pobedrya 1984) and (Bakhvalov & Panasenko 1989) is one of the most robust. The dynamic method has been used in piezoelectric-polymer compounds to calculate electromechanical coupling factors, yielding results that are closer to the experimental values than those predicted by homogenization methods (Zhang & Geng 1994, 614). In this work, we propose the comparison results obtained from both methods.

System studied

Let us consider a heterogeneous piezoelectric/piezomagnetic material (Fig. 1), made of alternating plates of piezoelectric and piezomagnetic materials, forming a parallel arrangement in the direction x_1, which is known as a composite material of the type 2-2. The coordinate system is chosen such that the x_3 axis is along the polarization direction of the piezoelectric and the piezomagnetic medium, the x_1 axis is perpendicular to the interfaces; therefore the discontinuity direction is in the x_1 direction and the x_2 axis is a long the plane of the plate.

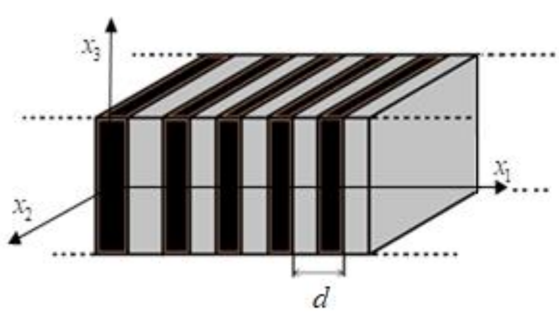

Fig. 1. Composite Scheme.

For a typical 2-2 composite, the dimensions are much larger than the period d and thickness. In this approximation, they can be considered as infinite. Thus, the problem is independent of the x_2 direction. The governing equations for the dynamic heterogeneous plates are

$$\frac{\partial \sigma_1}{\partial x_1}+\frac{\partial \sigma_5}{\partial x_3}=\rho\frac{\partial^2 u_1}{\partial t^2}, \qquad \frac{\partial \sigma_5}{\partial x_1}+\frac{\partial \sigma_3}{\partial x_3}=\rho\frac{\partial^2 u_3}{\partial t^2},$$
$$\frac{\partial D_1}{\partial x_1}+\frac{\partial D_3}{\partial x_3}=0, \qquad \frac{\partial B_1}{\partial x_1}+\frac{\partial B_3}{\partial x_3}=0, \tag{1}$$

where σ_I (we use the Voigt notation) are the components of stress tensor, u_i are the components of displacement vector, D_i are the components of electric displacement vector and B_i are the components of magnetic field vector.

The constitutive equations, which relate σ_I, D_i, B_i the components of the strain tensor S_I where $S_I = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$, the component of electric field intensity vector E_i and the components of magnetic field intensity vector H_i are given by:

$$\begin{aligned}
\sigma_1 &= c_{11}S_1 + c_{13}S_3 - e_{31}E_3 - q_{31}H_3,\\
\sigma_3 &= c_{13}S_1 + c_{33}S_3 - e_{33}E_3 - q_{33}H_3,\\
\sigma_5 &= 2c_{55}S_5 - e_{15}E_1 + q_{15}H_1,\\
D_1 &= 2e_{15}S_5 - \varepsilon_{11}E_1 - \lambda_{11}H_1,\\
D_3 &= e_{33}S_3 + e_{31}S_1 - \varepsilon_{33}E_3 - \lambda_{33}H_3,\\
B_1 &= 2q_{15}S_5 - \lambda_{11}E_1 - \mu_{11}H_1,\\
B_3 &= q_{33}S_3 + q_{31}S_1 - \lambda_{33}E_3 - \mu_{33}H_3.
\end{aligned} \tag{2}$$

The constitutive equations, which relate σ_I, D_i, B_i with , u_i the electric potential, φ and the magnetic potential ψ, are given by:

$$\sigma_1 = c_{11}\frac{\partial u_1}{\partial x_1} + c_{13}\frac{\partial u_3}{\partial x_3} + e_{31}\frac{\partial \varphi}{\partial x_3} + q_{31}\frac{\partial \psi}{\partial x_3},$$
$$\sigma_3 = c_{13}\frac{\partial u_1}{\partial x_1} + c_{33}\frac{\partial u_3}{\partial x_3} + e_{33}\frac{\partial \varphi}{\partial x_3} + q_{33}\frac{\partial \psi}{\partial x_3},$$
$$\sigma_5 = c_{55}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) + e_{15}\frac{\partial \varphi}{\partial x_1} + q_{15}\frac{\partial \psi}{\partial x_1},$$
$$D_1 = e_{15}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) - \varepsilon_{11}\frac{\partial \varphi}{\partial x_1} - \lambda_{11}\frac{\partial \psi}{\partial x_1}, \qquad (3)$$
$$D_3 = e_{33}\frac{\partial u_3}{\partial x_3} + e_{31}\frac{\partial u_1}{\partial x_1} - \varepsilon_{33}\frac{\partial \varphi}{\partial x_3} - \lambda_{33}\frac{\partial \psi}{\partial x_3},$$
$$B_1 = q_{15}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) - \lambda_{11}\frac{\partial \varphi}{\partial x_1} - \mu_{11}\frac{\partial \psi}{\partial x_1},$$
$$B_3 = q_{33}\frac{\partial u_3}{\partial x_3} + q_{31}\frac{\partial u_1}{\partial x_1} - \lambda_{33}\frac{\partial \varphi}{\partial x_3} - \mu_{33}\frac{\partial \psi}{\partial x_3},$$

Where we have used the quasi-static approximation of the fields. The symbols c_{ij}, ε_{ij}, μ_{ij}, λ_{ij}, e_{ij} and q_{ij} represent the elasticity, dielectric permittivity, magnetic permittivity, magnetoelectric, piezoelectric, and piezomagnetic tensors, respectively.

Homogeneous asymptotic method

The double-scale asymptotic homogenization (MHA) method introduces two spatial coordinate systems. The position of the body is denoted by the Cartesian coordinate system $\boldsymbol{X} = (x_1, x_2, x_3)$. We introduce the local variable $\boldsymbol{Y} = (y_1, y_2, y_3)$ whose components are given by $y_i = x_i/\alpha$; with $\alpha << 1$. The material functions are periodic respect to $\boldsymbol{Y}$.

An asymptotic double scale development around the parameter α for the displacement and for the potentials is proposed as follows

$$\begin{aligned} u_1(x_1, x_3, \mathrm{y}_1, t) &= u_1^{(0)}(x_1, x_3, t) + \alpha u_1^{(1)}(x_1, x_3, \mathrm{y}_1, t) + \cdots \\ u_3(x_1, x_3, \mathrm{y}_1, t) &= u_3^{(0)}(x_1, x_3, t) + \alpha u_3^{(1)}(x_1, x_3, \mathrm{y}_1, t) + \cdots \\ \varphi(x_1, x_3, \mathrm{y}_1, t) &= \varphi^{(0)}(x_1, x_3, t) + \alpha \varphi^{(1)}(x_1, x_3, \mathrm{y}_1, t) + \cdots \\ \psi(x_1, x_3, \mathrm{y}_1, t) &= \psi^{(0)}(x_1, x_3, t) + \alpha \psi^{(1)}(x_1, x_3, \mathrm{y}_1, t) + \cdots \end{aligned} \qquad (4)$$

Substituting eqs (4) into eqs (3), the constitutive equations take the following form:

$$\begin{aligned}
\sigma_1(x_1,x_3,\mathrm{y}_1,t) &= \sigma_1^{(0)}(x_1,x_3,\mathrm{y}_1,t)+\alpha\sigma_1^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\cdots\\
\sigma_3(x_1,x_3,\mathrm{y}_1,t) &= \sigma_3^{(0)}(x_1,x_3,\mathrm{y}_1,t)+\alpha\sigma_3^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\cdots\\
\sigma_5(x_1,x_3,\mathrm{y}_1,t) &= \sigma_5^{(0)}(x_1,x_3,\mathrm{y}_1,t)+\alpha\sigma_5^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\cdots\\
D_1(x_1,x_3,\mathrm{y}_1,t) &= D_1^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\alpha D_1^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\cdots\\
B_1(x_1,x_3,\mathrm{y}_1,t) &= B_1^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\alpha B_1^{(1)}(x_1,x_3,\mathrm{y}_1,t)+\cdots
\end{aligned}\tag{5}$$

Where,

$$\begin{aligned}
\sigma_1^{(k)} &= c_{11}\left(\frac{\partial u_1^{(k)}}{\partial x_1}+\frac{\partial u_1^{(k+1)}}{\partial y_1}\right)+c_{13}\left(\frac{\partial u_3^{(k)}}{\partial x_3}\right)+e_{31}\left(\frac{\partial \varphi^{(k)}}{\partial x_3}\right)+q_{31}\left(\frac{\partial \psi^{(k)}}{\partial x_3}\right),\\
\sigma_3^{(k)} &= c_{13}\left(\frac{\partial u_1^{(k)}}{\partial x_1}+\frac{\partial u_1^{(k+1)}}{\partial y_1}\right)+c_{33}\left(\frac{\partial u_3^{(k)}}{\partial x_3}\right)+e_{33}\left(\frac{\partial \varphi^{(k)}}{\partial x_3}\right)+q_{33}\left(\frac{\partial \psi^{(k)}}{\partial x_3}\right),\\
\sigma_5^{(k)} &= c_{55}\left(\left(\frac{\partial u_1^{(k)}}{\partial x_3}\right)+\left(\frac{\partial u_3^{(k)}}{\partial x_1}+\frac{\partial u_3^{(k+1)}}{\partial y_1}\right)\right)+e_{15}\left(\frac{\partial \varphi^{(k)}}{\partial x_1}+\frac{\partial \varphi^{(k+1)}}{\partial y_1}\right)+q_{15}\left(\frac{\partial \psi^{(k)}}{\partial x_1}+\frac{\partial \psi^{(k+1)}}{\partial y_1}\right).\\
D_1^{(k)} &= e_{15}\left(\left(\frac{\partial u_1^{(k)}}{\partial x_3}\right)+\left(\frac{\partial u_3^{(k)}}{\partial x_1}+\frac{\partial u_3^{(k+1)}}{\partial y_1}\right)\right)-\varepsilon_{11}\left(\frac{\partial \varphi^{(k)}}{\partial x_1}+\frac{\partial \varphi^{(k+1)}}{\partial y_1}\right)-\lambda_{11}\left(\frac{\partial \psi^{(k)}}{\partial x_1}+\frac{\partial \psi^{(k+1)}}{\partial y_1}\right),\\
D_3^{(k)} &= e_{33}\left(\frac{\partial u_3^{(k)}}{\partial x_3}\right)+e_{31}\left(\frac{\partial u_1^{(k)}}{\partial x_1}+\frac{\partial u_1^{(k+1)}}{\partial y_1}\right)-\varepsilon_{33}\left(\frac{\partial \varphi^{(k)}}{\partial x_3}\right)-\lambda_{33}\left(\frac{\partial \psi^{(k)}}{\partial x_3}\right),\\
B_1^{(k)} &= q_{15}\left(\left(\frac{\partial u_1^{(k)}}{\partial x_3}\right)+\left(\frac{\partial u_3^{(k)}}{\partial x_1}+\frac{\partial u_3^{(k+1)}}{\partial y_1}\right)\right)-\lambda_{11}\left(\frac{\partial \varphi^{(k)}}{\partial x_1}+\frac{\partial \varphi^{(k+1)}}{\partial y_1}\right)-\mu_{11}\left(\frac{\partial \psi^{(k)}}{\partial x_1}+\frac{\partial \psi^{(k+1)}}{\partial y_1}\right),\\
B_3^{(k)} &= q_{33}\left(\frac{\partial u_3^{(k)}}{\partial x_3}\right)+q_{31}\left(\frac{\partial u_1^{(k)}}{\partial x_1}+\frac{\partial u_1^{(k+1)}}{\partial y_1}\right)-\lambda_{33}\left(\frac{\partial \varphi^{(k)}}{\partial x_3}\right)-\mu_{33}\left(\frac{\partial \psi^{(k)}}{\partial x_3}\right).
\end{aligned}\tag{6}$$

Substituting eqs (5) and (6) in eqs (1) and rearranging terms of equal potential of α we have:

for α^{-1}

$$\frac{\partial \sigma_1^{(0)}}{\partial y_1} = 0,$$
$$\frac{\partial D_1^{(0)}}{\partial y_1} = 0, \qquad (7)$$
$$\frac{\partial B_1^{(0)}}{\partial y_1} = 0.$$

For $\alpha = 0$

$$\frac{\partial \sigma_1^{(0)}}{\partial x_1} + \frac{\partial \sigma_5^{(0)}}{\partial x_3} + \frac{\partial \sigma_1^{(1)}}{\partial y_1} = \rho \frac{\partial^2 u_1^{(0)}}{\partial t^2},$$
$$\frac{\partial \sigma_5^{(0)}}{\partial x_1} + \frac{\partial \sigma_3^{(0)}}{\partial x_3} + \frac{\partial \sigma_5^{(1)}}{\partial y_1} = \rho \frac{\partial^2 u_3^{(0)}}{\partial t^2},$$
$$\frac{\partial D_1^{(0)}}{\partial x_1} + \frac{\partial D_3^{(0)}}{\partial x_3} + \frac{\partial D_1^{(1)}}{\partial y_1} = 0, \qquad (8)$$
$$\frac{\partial B_1^{(0)}}{\partial x_1} + \frac{\partial B_3^{(0)}}{\partial x_3} + \frac{\partial B_1^{(1)}}{\partial y_1} = 0.$$

Taking the average per unit length $\left(\langle F \rangle = \frac{1}{|\mathbf{Y}|} \int F dy \right)$ of the expressions given by eqs(8) and using the periodicity of $\sigma^{(1)}$, $D^{(1)}$ and $B^{(1)}$ with respect to $\mathbf{Y}$, the dynamic expressions of the homogeneous problem is obtained as follows:

$$\frac{\partial \bar{\sigma}_1^{(0)}}{\partial x_1} + \frac{\partial \bar{\sigma}_5^{(0)}}{\partial x_3} = \bar{\rho} \frac{\partial^2 u_1^{(0)}}{\partial t^2},$$
$$\frac{\partial \bar{\sigma}_5^{(0)}}{\partial x_1} + \frac{\partial \bar{\sigma}_3^{(0)}}{\partial x_3} = \bar{\rho} \frac{\partial^2 u_3^{(0)}}{\partial t^2},$$
$$\frac{\partial \bar{D}_1^{(0)}}{\partial x_1} + \frac{\partial \bar{D}_3^{(0)}}{\partial x_3} = 0, \qquad (9)$$
$$\frac{\partial \bar{B}_1^{(0)}}{\partial x_1} + \frac{\partial \bar{B}_3^{(0)}}{\partial x_3} = 0,$$

where $\bar{F}$ is the average of F (global valor of F). Let N, W, S, Φ, Θ, Ξ, Ψ, Ω and Υ be auxiliary functions that only depend on the local variables (local functions) and periodicty in **Y.** We can write $u_1^{(1)}$, $u_3^{(1)}$, $\varphi^{(1)}$ and $\psi^{(1)}$ in terms of the local functions as follows:

$$
\begin{aligned}
u_n^{(1)} &= N_n^{kl}\left(y_1\right)\frac{\partial u_k^{(0)}}{\partial x_l} + W_n^l\left(y_1\right)\frac{\partial \varphi^{(0)}}{\partial x_l} + S_n^l\left(y_1\right)\frac{\partial \psi^{(0)}}{\partial x_l},\\
\varphi^{(1)} &= \Phi^{kl}\left(y_1\right)\frac{\partial u_k^{(0)}}{\partial x_l} + \Theta^l\left(y_1\right)\frac{\partial \varphi^{(0)}}{\partial x_l} + \Xi^l\left(y_1\right)\frac{\partial \psi^{(0)}}{\partial x_l},\\
\psi^{(1)} &= \Psi^{kl}\left(y_1\right)\frac{\partial u_k^{(0)}}{\partial x_l} + \Omega^l\left(y_1\right)\frac{\partial \varphi^{(0)}}{\partial x_l} + \Upsilon^l\left(y_1\right)\frac{\partial \psi^{(0)}}{\partial x_l},
\end{aligned}
\tag{10}
$$

Where Eintein's sum has been used.

Substituting the expressions given by eqs (9) in eqs (5) and taking the average, seven equations are obtained which relate the averages of zero order of the fields, with the derivatives of the components of zero order of the displacements and the potentials. These equations are shown in appendix A and they are numerated as (A1). Comparing the equations in Annex (A1) with the contiguous equations, it can be seen that the coefficients that multiply the derivatives are the effective coefficients. These expressions are the constitutive equations of the homogeneous problem. The coefficients that appear in equations (A1) must be independent of $\mathbf{Y}$, *i.e*. their derivatives with respect to y_1 must be equal to zero. From this condition, three systems of equations are obtained as follows:

$$
L_1=\begin{cases}
\dfrac{d\sigma_1\left(N_n^{kk},\Phi^{kk},\Psi^{kk}\right)}{dy_1}=-\dfrac{dc_{1k}}{dy_1}\\
\dfrac{d\sigma_5\left(N_n^{13},\Phi^{13},\Psi^{13}\right)}{dy_1}=-\dfrac{dc_{55}}{dy_1}\\
\dfrac{dD_1\left(N_n^{13},\Phi^{13},\Psi^{13}\right)}{dy_1}=-\dfrac{de_{15}}{dy_1}\\
\dfrac{dB_1\left(N_n^{13},\Phi^{13},\Psi^{13}\right)}{dy_1}=-\dfrac{dq_{15}}{dy_1}
\end{cases}
\quad
L_2=\begin{cases}
\dfrac{d\sigma_1\left(W_n^{3},\Theta^{3},\Omega^{3}\right)}{dy_1}=-\dfrac{de_{31}}{dy_1}\\
\dfrac{d\sigma_5\left(W_n^{1},\Theta^{1},\Omega^{1}\right)}{dy_1}=-\dfrac{de_{15}}{dy_1}\\
\dfrac{dD_1\left(W_n^{1},\Theta^{1},\Omega^{1}\right)}{dy_1}=\dfrac{d\varepsilon_{11}}{dy_1}\\
\dfrac{dB_1\left(W_n^{1},\Theta^{1},\Omega^{1}\right)}{dy_1}=\dfrac{d\lambda_{11}}{dy_1}
\end{cases}
\quad
L_3=\begin{cases}
\dfrac{d\sigma_1\left(S_n^{3},\Xi^{3},\Upsilon^{3}\right)}{dy_1}=-\dfrac{dq_{31}}{dy_1}\\
\dfrac{d\sigma_5\left(S_n^{1},\Xi^{1},\Upsilon^{1}\right)}{dy_1}=-\dfrac{dq_{15}}{dy_1}\\
\dfrac{dD_1\left(S_n^{1},\Xi^{1},\Upsilon^{1}\right)}{dy_1}=\dfrac{d\lambda_{11}}{dy_1}\\
\dfrac{dB_1\left(S_n^{1},\Xi^{1},\Upsilon^{1}\right)}{dy_1}=\dfrac{d\mu_{11}}{dy_1}
\end{cases}
\tag{11}
$$

Wich can be solved by using the periodicity of the local functions. In this way the effective coefficients can be obtained through the following relations:

$$\text{Elastic}\begin{cases}\bar{c}_{11}=\left\langle c_{11}^{-1}\right\rangle^{-1},\\ \bar{c}_{13}=\left\langle c_{11}^{-1}c_{13}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1},\\ \bar{c}_{33}=\left\langle c_{33}\right\rangle+\left\langle c_{11}^{-1}c_{13}^{2}\right\rangle-\left\langle c_{11}^{-1}c_{13}\right\rangle^{2}\left\langle c_{11}^{-1}\right\rangle^{-1},\\ \bar{c}_{55}=M_{11}.\end{cases} \tag{12}$$

$$\text{Piezoelectric}\begin{cases}\bar{e}_{31}=\left\langle e_{31}c_{11}^{-1}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1},\\ \bar{e}_{33}=\left\langle e_{33}\right\rangle+\left\langle e_{31}c_{11}^{-1}\right\rangle\left\langle c_{11}^{-1}c_{13}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1}-\left\langle e_{31}c_{13}c_{11}^{-1}\right\rangle,\\ \bar{e}_{55}=M_{12}.\end{cases} \tag{13}$$

$$\text{Piezomagnetic}\begin{cases}\bar{q}_{31}=\left\langle q_{31}c_{11}^{-1}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1},\\ \bar{q}_{33}=\left\langle q_{33}\right\rangle+\left\langle q_{31}c_{11}^{-1}\right\rangle\left\langle c_{11}^{-1}c_{13}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1}-\left\langle q_{31}c_{13}c_{11}^{-1}\right\rangle,\\ \bar{q}_{55}=M_{13}.\end{cases} \tag{14}$$

$$\text{Dielectric}\begin{cases}\bar{\varepsilon}_{11}=-M_{22},\\ \bar{\varepsilon}_{33}=\left\langle \varepsilon_{33}\right\rangle-\left\langle e_{31}^{2}c_{11}^{-1}\right\rangle+\left\langle e_{31}c_{11}^{-1}\right\rangle^{2}\left\langle c_{11}^{-1}\right\rangle^{-1}.\end{cases} \tag{15}$$

$$\text{Diamagnetic}\begin{cases}\bar{\mu}_{11}=-M_{33},\\ \bar{\mu}_{33}=\left\langle \mu_{33}\right\rangle-\left\langle q_{31}^{2}c_{11}^{-1}\right\rangle+\left\langle q_{31}c_{11}^{-1}\right\rangle^{2}\left\langle c_{11}^{-1}\right\rangle^{-1}.\end{cases} \tag{16}$$

$$\text{Magnetoelectric}\begin{cases}\bar{\lambda}_{11}=-M_{23},\\ \bar{\lambda}_{33}=\left\langle \lambda_{33}\right\rangle-\left\langle q_{31}e_{31}c_{11}^{-1}\right\rangle+\left\langle q_{31}c_{11}^{-1}\right\rangle\left\langle e_{31}c_{11}^{-1}\right\rangle\left\langle c_{11}^{-1}\right\rangle^{-1}.\end{cases} \tag{17}$$

Where $M=\left(\left\langle N\right\rangle^{-1}\right)^{-1}$ and $N=\begin{pmatrix}c_{55} & e_{15} & q_{15}\\ e_{15} & -\varepsilon_{11} & -\lambda_{11}\\ q_{15} & -\lambda_{11} & -\mu_{11}\end{pmatrix}.$

Fig. 2 shows the magnetoelectric coefficients, which have the most interesting behavior.

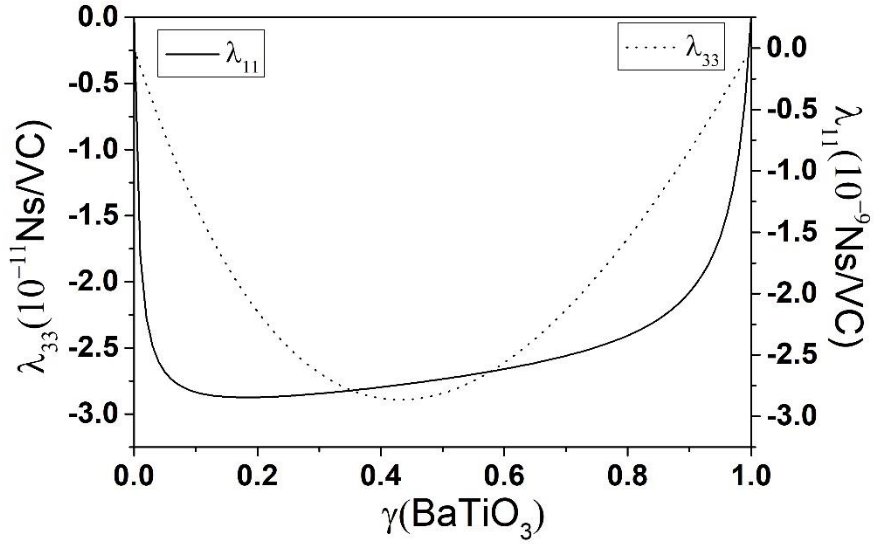

Fig. 2. Magnetoelectric effective coefficients of periodic multilayer of $BaTiO_3$ and $CoFe_2O_4$.

Coupling magnetoelectric constant

The thermodynamic potential W is defined from the internal energy U (Pérez-Fernández, 209, 343):

$$W = U - E_i D_i - H_i B_i. \tag{18}$$

Internal energy is defined as

$$U = \tfrac{1}{2}\sigma_I S_I + \tfrac{1}{2}E_i D_i + \tfrac{1}{2}H_i B_i. \tag{19}$$

Substituting eqs (19) and the constitutive equations (3), into (18) the following expression are obtained:

$$\begin{aligned} U &= \tfrac{1}{2}S_I c_{IJ} S_I - \tfrac{1}{2}E_i \varepsilon_{ij} E_j - \tfrac{1}{2}H_i \mu_{ij} H_j - S_I e_{Ij} E_j - S_I q_{Ij} H_j - E_i \lambda_{ij} H_j, \\ &= W_c - W_e - W_q - 2W_{ce} - 2W_{cq} - 2W_{eq}, \end{aligned} \tag{20}$$

Where $W_c = \frac{1}{2} S_I c_{IJ} S_I$ is the elastic energy density, $W_e = \frac{1}{2} E_i \varepsilon_{ij} E_j$ is the electric energy density, $W_q = \frac{1}{2} H_i \mu_{ij} H_j$ is the magnetic energy density, $W_{ce} = \frac{1}{2} S_I e_{Ij} E_j$ is the piezoelectric energy density, $W_{cq} = \frac{1}{2} S_I q_{Ij} H_j$ is the piezomagetic energy density and $W_{eq} = \frac{1}{2} E_i \lambda_{ij} H_j$ is the magneto-eectric energy density.

The coupling ME constant in terms of this potential can written as

$$O_t = \frac{W_{ce}}{\sqrt{W_c W_e}}. \tag{21}$$

In this way, the coupling ME constant is obtained by this method.

Dynamical method

The second method, which we have called the dynamic method, is to study the behavior of the compound before the propagation of vertically polarized shear waves (SV). First the dispersion curves are obtained, and from them the coupling factor.

Combining (1) and (2) we have four differential equations of second order, which describe the behavior of the elastic displacements u_1, u_3 and the electric potential φ for the composite. The solution of these systems must be solved in each medium independently. The solution of the system is propose as plane wave for each medium, *i.e.*

$$\begin{aligned} u_3 &= A\exp\left(i\left(k_1x_1 + k_3x_3 - \omega t\right)\right), \\ u_1 &= B\exp\left(i\left(k_1x_1 + k_3x_3 - \omega t\right)\right), \\ \varphi &= C\exp\left(i\left(k_1x_1 + k_3x_3 - \omega t\right)\right), \\ \psi &= D\exp\left(i\left(k_1x_1 + k_3x_3 - \omega t\right)\right), \end{aligned} \tag{22}$$

Where, k_i are the components of the wave vector, ω is angular frequency and *A, B*, *C* and *D* are indeterminate constants. Let's work first in the piezoelectric medium.

Substituting (22) into the system we obtain three homogeneous equations with three unknown independent constants *A*, *B* and *C*. These equations can been writer in matrix form:

$$QCo^T = 0, \qquad (23)$$

Where $$Co = \left(A, \quad B, \quad C\right) \qquad (24)$$

And

$$Q = \begin{pmatrix} c_{33}k_3^2 + c_{44}k_1^2 - \rho\omega^2 & \left(c_{13} + c_{44}\right)k_1k_3 & e_{33}k_3^2 + e_{15}k_1^2 & 0 \\ \left(c_{13} + c_{44}\right)k_1k_3 & c_{11}k_1^2 + c_{44}k_3^2 - \rho\omega^2 & \left(e_{15} + e_{44}\right)k_1k_3 & 0 \\ e_{33}k_3^2 + e_{15}k_1^2 & \left(e_{15} + e_{44}\right)k_1k_3 & -\left(\varepsilon_{11}k_1^2 + \varepsilon_{33}k_3^2\right) & 0 \\ 0 & 0 & 0 & -\left(\mu_{11}k_1^2 + \mu_{33}k_3^2\right) \end{pmatrix}. \qquad (25)$$

The condition for a nontrivial solution is that the determinant of the coefficients vanished. In the piezoelectric medium this determinant can be write as:

$$\left|W\right| = 0. \qquad (26)$$

The expression (26) is an implicit function of ω, k_1 and k_3. For each pair $\left(\omega, k_3\right)$ four k_1 values are obtained, which correspond to the quasi-longitudinal, quasi-shear, quasi-piezoelectric and quasi-piezomagnetic wave. In the piezoelectric material the quasi-piezomagnetic wave is such that

$$k_1 = \frac{\mu_{33}}{\mu_{11}} ik_3 .$$

Due to the symmetry of the system (1), the solution for the case of the piezoelectric material can be written as (27) which is one of the two modes of the Lamb wave.

$$
\begin{aligned}
u_3^e &= \sum_{i=1}^{3} R_i^e \cos\left(k_1^{ie} x_1\right)\sin\left(k_3 x_3\right),\\
u_1^e &= \sum_{i=1}^{3} f_i^e R_i^e \sin\left(k_1^{ie} x_1\right)\cos\left(k_3 x_3\right),\\
\varphi^e &= \sum_{i=1}^{3} g_i^e R_i^e \cos\left(k_1^{ie} x_1\right)\sin\left(k_3 x_3\right),\\
\psi^e &= R_4^e \cosh\left(k_3 x_3\right)\sin\left(k_3 x_3\right),
\end{aligned}
\tag{27}
$$

where index *i* refers to each of the values of k_1^{ie}, index e indicate piezoelectric medium, f_i^e, g_i^e are obtained from the relationship between A, B and C. Since the system is not determined, its solution is indeterminate in at least one constant, R_i^e are this constant. Substituting (27) into (1) the form of the fields are obtained.

A similar development is carried out in the case of the piezomagnetic medium and similar solutions are obtained (28).

$$
\begin{aligned}
u_3^q &= \sum_{i=1}^{3} R_i^q \cos\left(k_1^{iq} x_1\right)\sin\left(k_3 x_3\right),\\
u_1^q &= \sum_{i=1}^{3} f_i^q R_i^q \sin\left(k_1^{iq} x_1\right)\cos\left(k_3 x_3\right),\\
\varphi^q &= R_4^q \cosh\left(\frac{\varepsilon_{33}}{\varepsilon_{11}} k_3 x_3\right)\sin\left(k_3 x_3\right),\\
\psi^q &= \sum_{i=1}^{3} g_i^q R_i^q \cos\left(k_1^{iq} x_1\right)\sin\left(k_3 x_3\right).
\end{aligned}
\tag{28}
$$

Where index q indicate piezomagnetic medium.

The contact conditions give the conditions to be able to solve the system. We consider condition ideal contact in the interphases, that is to say conditions of continuity, as shown in (29).

$$\begin{aligned} u_1^e &= u_1^q, & \sigma_1^e &= \sigma_1^q, \\ u_3^e &= u_3^q, & \sigma_5^e &= \sigma_5^q, \\ \varphi^e &= \varphi^q, & D_1^e &= D_1^q, \\ \psi^e &= \psi^q, & B_1^e &= B_1^q. \end{aligned} \tag{29}$$

These conditions are evaluated at the interphases $\left(x_1 = {}^{\gamma d}\!/_2\right.$ where γ is the volumetric fraction of $\left.BaTiO_3\right)$. They are eight homogenous equations with eight indeterminate constants $\left(R_i^e\right.$ and $\left.R_i^q \text{ whith } i = 1,2,3,4\right)$. The condition for a nontrivial solution is that the determinant of the matrix associated to the system vanished. This condition gives a family of implicit functions of k_3 and ω. These functions are the dispersion curves for the composite (**Fig. 3**).

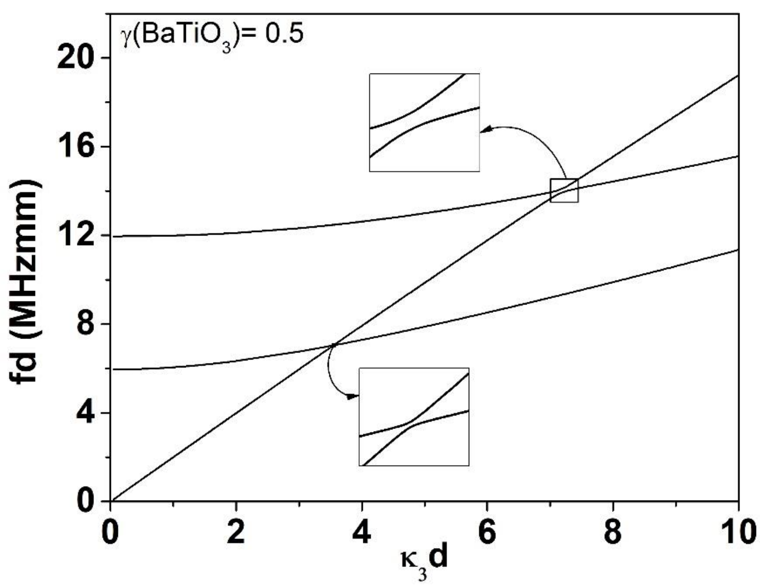

Fig. 3. Dispersion curves for a volumetric fraction of $BaTiO_3$ of 0.5.

Coupling magnetoelectric constant

To determine the coupling factor ME O_t using this model, have been used the definition of the coupling factor ME from the relationship between the wave velocity at *E* and *H* constants $\left(v^{EH}\right)$ and wave velocity at *D* and *B* constants $\left(v^{DB}\right)$.

$$\left(\frac{v^{EH}}{v^{DB}}\right)^2 = \left(1 - O_t^2\right). \tag{30}$$

For a single piezoelectric material, the velocity v^{EH} can be obtained from the dispersion curves as the slope of the first mode. For a composite material, this method is valid in the limit $k_3 \rightarrow 0$ as discussed in (Zhang & Geng 1994, 614). Similarly, v^{DB} is obtained but now only using the elastic equations. The procedure is also discussed in (Zhang & Geng 1994, 614).

Results

By means of the first method (the asymptotic homogeneous method) the ME coupling factor of piezoelectric/piezomagnetic composites with layered of $BaTiO_3$ and $CoFe_2O_4$ was computed starting from the effective coefficients of the composite. While the second method (dynamical method) uses the IEEE definition and determines the ME coupling factor trough the v^{EH} and v^{DB} obtained from the dispersion curves. Figure 4 shows the results obtained.

In **Fig. 4** the solid line represents the result obtained from the asymptotic homogenization method. As follows from the formulation of the method, this results is an analytical function. While for the dynamical method we obtained a discrete plot because the calculations have been made for each volumetric fraction (the results are shown by black squares). This is a disadvantage of this method; however, it has the advantage of making the calculations directly from the phase constants and not from the effective constants. In (Zhang & Geng 1994, 614) it is shown that this second method is closer to experimental results than an homogenization method for the calculation of k_t.

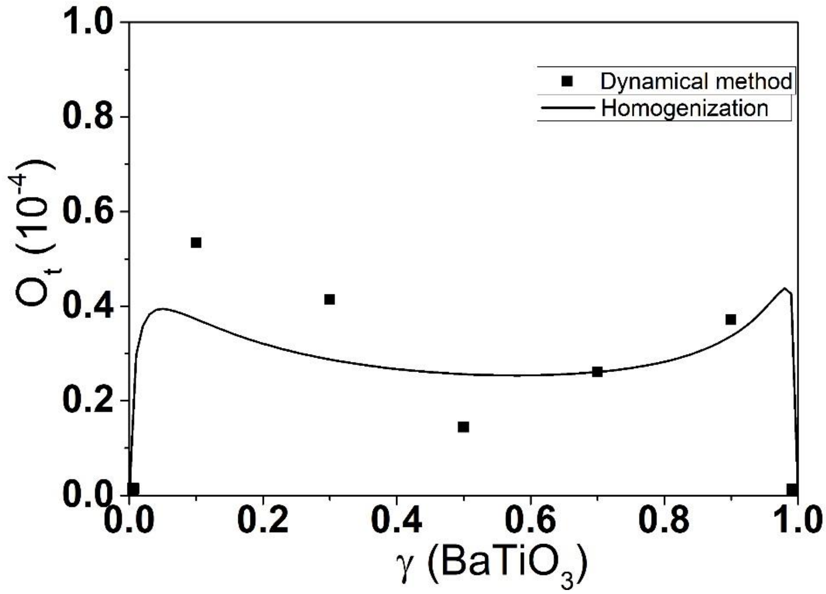

Fig. 4. ME coupling factor as a function of piezoelectric volumetric fraction obtained for both methods. Note that the continues line passes by the points (0,0) y (1,0), as it should be. Furthermore, there is a square in each of these two points, but they are indistinguishable because they overlap with the axes.

The results of these two methods show a very good agreement at low/high volumetric fractions of $\gamma\left(\mathrm{BaTiO_3}\right)$. Both approach to zero in the limit cases when one of the phases is not present. The ME effect is a second order effect that appears in the compound through the interaction of both phases. However, in the center part of the interval the results are different although they show a similar behavior. This result shows that both methods provide a guide for the manufacture of laminated materials showing a ME effect. This mismatch is also obtained by (Zhang & Gheng 1994, 614) in the calculation of an electromechanical coupling factor. They also obtained a greater co-presence in compounds with a larger amount of piezoelectric. They also demonstrated that the dynamic method out performs the results obtained through the homogenization methods when compared with the experimental results.

Homogenization methods constitute an approximation for modering heterogeneous materials as homogeneous materials. In order to make this approximation, strong conditions are required on the wavelengths which are used. The dynamical method has a better performance; however it may present numerical instabilities.

Conclusions

In this paper two methods to determine the ME coupling factor of piezoelectric-piezomagnetic multilaminates were used. The homogeneization method is based on calculations of the effective properties of the composite and from this method the effective coupling factor can be determined. The dynamic method, in which the ME coupling factor is obtained from determining the slope of the dispersion curves, was described. Despite the difference between both methods; a similar trend is observed in both calculations. These results provide a valid guide for building a device with ME properties.

Aknowledments

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Appendix A

$$\begin{aligned}
\bar{\sigma}_1^{(0)} &= \langle c_{11}+\tau_{11}\rangle \frac{\partial u_1^{(0)}}{\partial x_1} + \langle c_{13}+\tau_{13}\rangle \frac{\partial u_3^{(0)}}{\partial x_3} + \langle e_{31}+d_{31}\rangle \frac{\partial \varphi^{(0)}}{\partial x_3} + \langle q_{31}+h_{31}\rangle \frac{\partial \psi^{(0)}}{\partial x_3} \\
\bar{\sigma}_3^{(0)} &= \langle c_{13}+\tau_{13}\rangle \frac{\partial u_1^{(0)}}{\partial x_1} + \langle c_{33}+\tau_{33}\rangle \frac{\partial u_3^{(0)}}{\partial x_3} + \langle e_{33}+d_{33}\rangle \frac{\partial \varphi^{(0)}}{\partial x_3} + \langle q_{33}+h_{33}\rangle \frac{\partial \psi^{(0)}}{\partial x_3} \\
\bar{\sigma}_5^{(0)} &= \langle c_{55}+\tau_{55}\rangle \left(\frac{\partial u_1^{(0)}}{\partial x_3} + \frac{\partial u_3^{(0)}}{\partial x_1}\right) + \langle e_{15}+d_{15}\rangle \frac{\partial \varphi^{(0)}}{\partial x_1} + \langle q_{15}+h_{15}\rangle \frac{\partial \psi^{(0)}}{\partial x_1} \\
\bar{D}_1^{(0)} &= \langle e_{15}+\zeta_{15}\rangle \left(\frac{\partial u_1^{(0)}}{\partial x_3} + \frac{\partial u_3^{(0)}}{\partial x_1}\right) - \langle \varepsilon_{11}-\delta_{11}\rangle \frac{\partial \varphi^{(0)}}{\partial x_1} - \langle \lambda_{11}-\beta_{11}\rangle \frac{\partial \psi^{(0)}}{\partial x_1} \\
\bar{D}_3^{(0)} &= \langle e_{33}+\zeta_{33}\rangle \frac{\partial u_3^{(0)}}{\partial x_3} + \langle e_{31}+\zeta_{31}\rangle \frac{\partial u_1^{(0)}}{\partial x_1} - \langle \varepsilon_{33}-\delta_{33}\rangle \frac{\partial \varphi^{(0)}}{\partial x_3} - \langle \lambda_{33}-\beta_{33}\rangle \frac{\partial \psi^{(0)}}{\partial x_3} \\
\bar{B}_1^{(0)} &= \langle q_{15}+\xi_{15}\rangle \left(\frac{\partial u_1^{(0)}}{\partial x_3} + \frac{\partial u_3^{(0)}}{\partial x_1}\right) - \langle \lambda_{11}-\kappa_{11}\rangle \frac{\partial \varphi^{(0)}}{\partial x_1} - \langle \mu_{11}-\chi_{11}\rangle \frac{\partial \psi^{(0)}}{\partial x_1} \\
\bar{B}_3^{(0)} &= \langle q_{33}+\xi_{33}\rangle \frac{\partial u_3^{(0)}}{\partial x_3} + \langle q_{31}+\xi_{31}\rangle \frac{\partial u_1^{(0)}}{\partial x_1} - \langle \lambda_{33}-\kappa_{33}\rangle \frac{\partial \varphi^{(0)}}{\partial x_3} - \langle \mu_{33}-\chi_{33}\rangle \frac{\partial \psi^{(0)}}{\partial x_3},
\end{aligned} \tag{A1}$$

Where

$$
\begin{cases}
\tau_{11} = c_{11}\dfrac{\partial N_1^{11}}{\partial y_1} + c_{13}\dfrac{\partial N_3^{11}}{\partial y_3} + e_{31}\dfrac{\partial \Phi^{11}}{\partial y_3} + q_{31}\dfrac{\partial \Psi^{11}}{\partial y_3} \\
\tau_{33} = c_{13}\dfrac{\partial N_1^{33}}{\partial y_1} + c_{33}\dfrac{\partial N_3^{33}}{\partial y_3} + e_{33}\dfrac{\partial \Phi^{33}}{\partial y_3} + q_{33}\dfrac{\partial \Psi^{33}}{\partial y_3} \\
\tau_{13} = c_{11}\dfrac{\partial N_1^{33}}{\partial y_1} + c_{13}\dfrac{\partial N_3^{33}}{\partial y_3} + e_{31}\dfrac{\partial \Phi^{33}}{\partial y_3} + q_{31}\dfrac{\partial \Psi^{33}}{\partial y_3} \\
\tau_{55} = c_{55}\left(\dfrac{\partial N_3^{13}}{\partial y_1} + \dfrac{\partial N_1^{13}}{\partial y_3}\right) + e_{15}\dfrac{\partial \Phi^{13}}{\partial y_1} + q_{15}\dfrac{\partial \Psi^{13}}{\partial y_1} \\
d_{15} = e_{15}\left(\dfrac{\partial N_3^{13}}{\partial y_1} + \dfrac{\partial N_1^{13}}{\partial y_3}\right) - \varepsilon_{11}\dfrac{\partial \Phi^{13}}{\partial y_1} - \lambda_{11}\dfrac{\partial \Psi^{13}}{\partial y_1} \\
d_{31} = e_{31}\dfrac{\partial N_1^{11}}{\partial y_1} + e_{33}\dfrac{\partial N_3^{11}}{\partial y_3} - \varepsilon_{33}\dfrac{\partial \Phi^{11}}{\partial y_3} - \lambda_{33}\dfrac{\partial \Psi^{11}}{\partial y_3} \\
d_{33} = e_{31}\dfrac{\partial N_1^{33}}{\partial y_1} + e_{33}\dfrac{\partial N_3^{33}}{\partial y_3} - \varepsilon_{33}\dfrac{\partial \Phi^{33}}{\partial y_3} - \lambda_{33}\dfrac{\partial \Psi^{33}}{\partial y_3} \\
h_{15} = q_{15}\left(\dfrac{\partial N_3^{13}}{\partial y_1} + \dfrac{\partial N_1^{13}}{\partial y_3}\right) - \lambda_{11}\dfrac{\partial \Phi^{13}}{\partial y_1} - \mu_{11}\dfrac{\partial \Psi^{13}}{\partial y_1} \\
h_{31} = q_{31}\dfrac{\partial N_1^{11}}{\partial y_1} + q_{33}\dfrac{\partial N_3^{11}}{\partial y_3} - \lambda_{33}\dfrac{\partial \Phi^{11}}{\partial y_3} - \mu_{33}\dfrac{\partial \Psi^{11}}{\partial y_3} \\
h_{33} = q_{31}\dfrac{\partial N_1^{33}}{\partial y_1} + q_{33}\dfrac{\partial N_3^{33}}{\partial y_3} - \lambda_{33}\dfrac{\partial \Phi^{33}}{\partial y_3} - \mu_{33}\dfrac{\partial \Psi^{33}}{\partial y_3}
\end{cases}
\qquad (A2)
$$

$$\begin{cases}
\zeta_{31} = c_{11}\dfrac{\partial W_1^3}{\partial y_1} + c_{13}\dfrac{\partial W_3^3}{\partial y_3} + e_{31}\dfrac{\partial \Theta^3}{\partial y_3} + q_{31}\dfrac{\partial \Omega^3}{\partial y_3} \\
\zeta_{33} = c_{13}\dfrac{\partial W_1^3}{\partial y_1} + c_{33}\dfrac{\partial W_3^3}{\partial y_3} + e_{33}\dfrac{\partial \Theta^3}{\partial y_3} + q_{33}\dfrac{\partial \Omega^3}{\partial y_3} \\
\zeta_{15} = c_{55}\left(\dfrac{\partial W_3^1}{\partial y_1} + \dfrac{\partial W_1^1}{\partial y_3}\right) + e_{15}\dfrac{\partial \Theta^1}{\partial y_1} + q_{15}\dfrac{\partial \Omega^1}{\partial y_1} \\
\delta_{11} = e_{15}\left(\dfrac{\partial W_3^1}{\partial y_1} + \dfrac{\partial W_1^1}{\partial y_3}\right) - \varepsilon_{11}\dfrac{\partial \Theta^1}{\partial y_1} - \lambda_{11}\dfrac{\partial \Omega^1}{\partial y_1} \\
\delta_{33} = e_{31|}\dfrac{\partial W_1^3}{\partial y_1} + e_{33}\dfrac{\partial W_3^3}{\partial y_3} - \varepsilon_{33}\dfrac{\partial \Theta^3}{\partial y_3} - \lambda_{33}\dfrac{\partial \Omega^3}{\partial y_3} \\
\beta_{11} = q_{15}\left(\dfrac{\partial W_3^1}{\partial y_1} + \dfrac{\partial W_1^1}{\partial y_3}\right) - \lambda_{11}\dfrac{\partial \Theta^1}{\partial y_1} - \mu_{11}\dfrac{\partial \Omega^{13}}{\partial y_1} \\
\beta_{33} = q_{31|}\dfrac{\partial W_1^3}{\partial y_1} + q_{33}\dfrac{\partial W_3^3}{\partial y_3} - \lambda_{33}\dfrac{\partial \Theta^3}{\partial y_3} - \mu_{33}\dfrac{\partial \Omega^3}{\partial y_3}
\end{cases} \qquad (A3)$$

$$\begin{cases}
\xi_{31} = c_{11}\dfrac{\partial S_1^3}{\partial y_1} + c_{13}\dfrac{\partial S_3^3}{\partial y_3} + e_{31}\dfrac{\partial \Xi^3}{\partial y_3} + q_{31}\dfrac{\partial \Upsilon^3}{\partial y_3} \\
\xi_{33} = c_{13}\dfrac{\partial S_1^3}{\partial y_1} + c_{33}\dfrac{\partial S_3^3}{\partial y_3} + e_{33}\dfrac{\partial \Xi^3}{\partial y_3} + q_{33}\dfrac{\partial \Upsilon^3}{\partial y_3} \\
\xi_{15} = c_{55}\left(\dfrac{\partial S_3^1}{\partial y_1} + \dfrac{\partial S_1^1}{\partial y_3}\right) + e_{15}\dfrac{\partial \Xi^1}{\partial y_1} + q_{15}\dfrac{\partial \Upsilon^1}{\partial y_1} \\
\kappa_{11} = e_{15}\left(\dfrac{\partial S_3^1}{\partial y_1} + \dfrac{\partial S_1^1}{\partial y_3}\right) - \varepsilon_{11}\dfrac{\partial \Xi^1}{\partial y_1} - \lambda_{11}\dfrac{\partial \Upsilon^1}{\partial y_1} \\
\kappa_{33} = e_{31|}\dfrac{\partial S_1^3}{\partial y_1} + e_{33}\dfrac{\partial S_3^3}{\partial y_3} - \varepsilon_{33}\dfrac{\partial \Xi^3}{\partial y_3} - \lambda_{33}\dfrac{\partial \Upsilon^3}{\partial y_3} \\
\chi_{11} = q_{15}\left(\dfrac{\partial S_3^1}{\partial y_1} + \dfrac{\partial S_1^1}{\partial y_3}\right) - \lambda_{11}\dfrac{\partial \Xi^1}{\partial y_1} - \mu_{11}\dfrac{\partial \Upsilon^{13}}{\partial y_1} \\
\chi_{33} = q_{31|}\dfrac{\partial S_1^3}{\partial y_1} + q_{33}\dfrac{\partial S_3^3}{\partial y_3} - \lambda_{33}\dfrac{\partial \Xi^3}{\partial y_3} - \mu_{33}\dfrac{\partial \Upsilon^3}{\partial y_3}
\end{cases} \qquad (A4)$$